Lecture 35

More NP-complete Problems

NP-Completeness and NP-Hardness

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Transitivity in Reduction: If $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.

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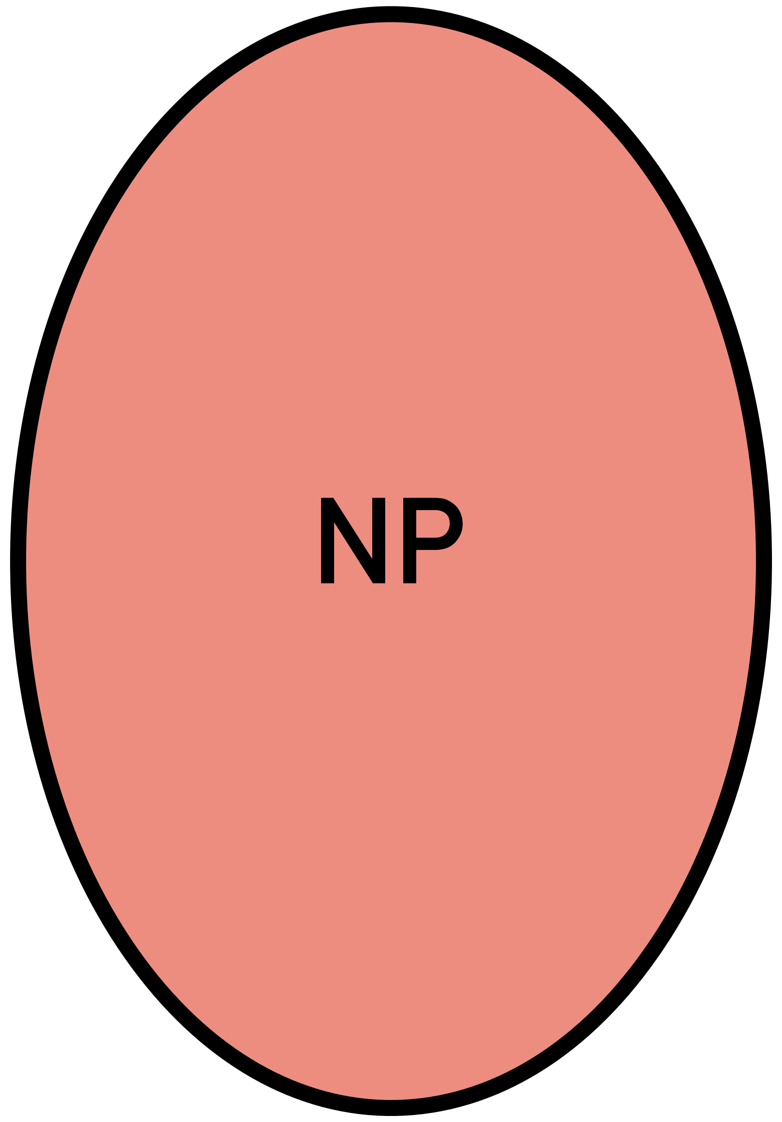

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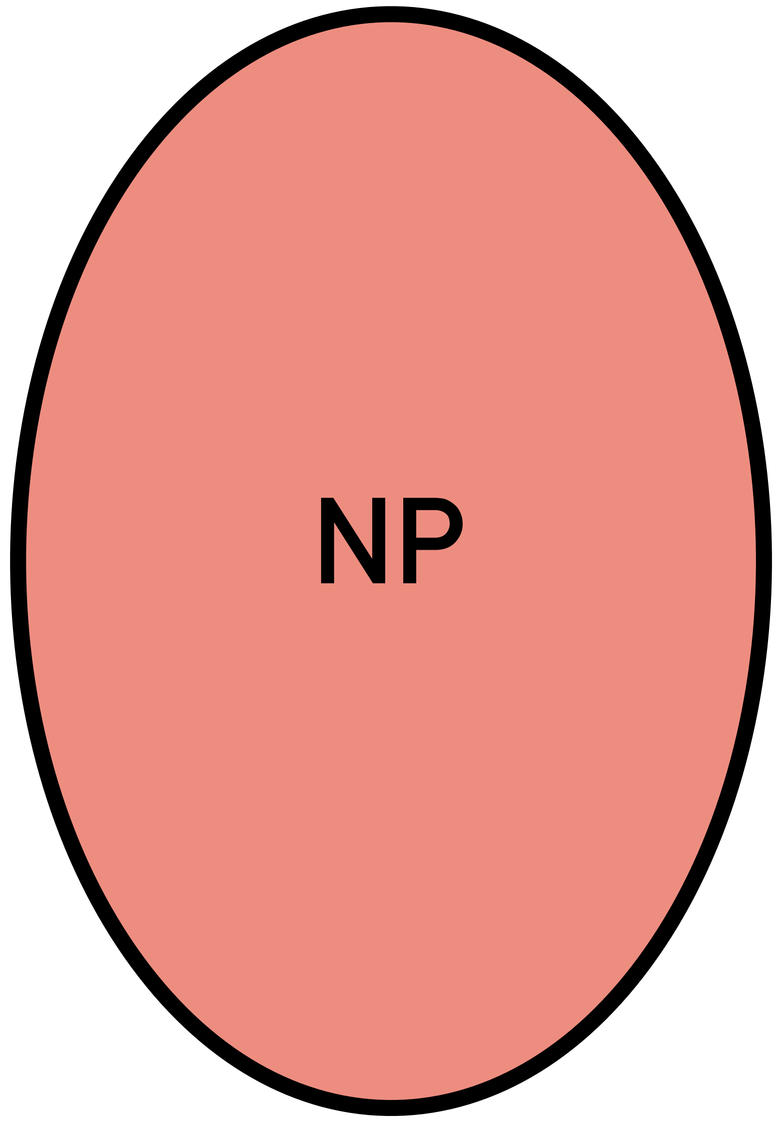

$\leq_p \quad L$

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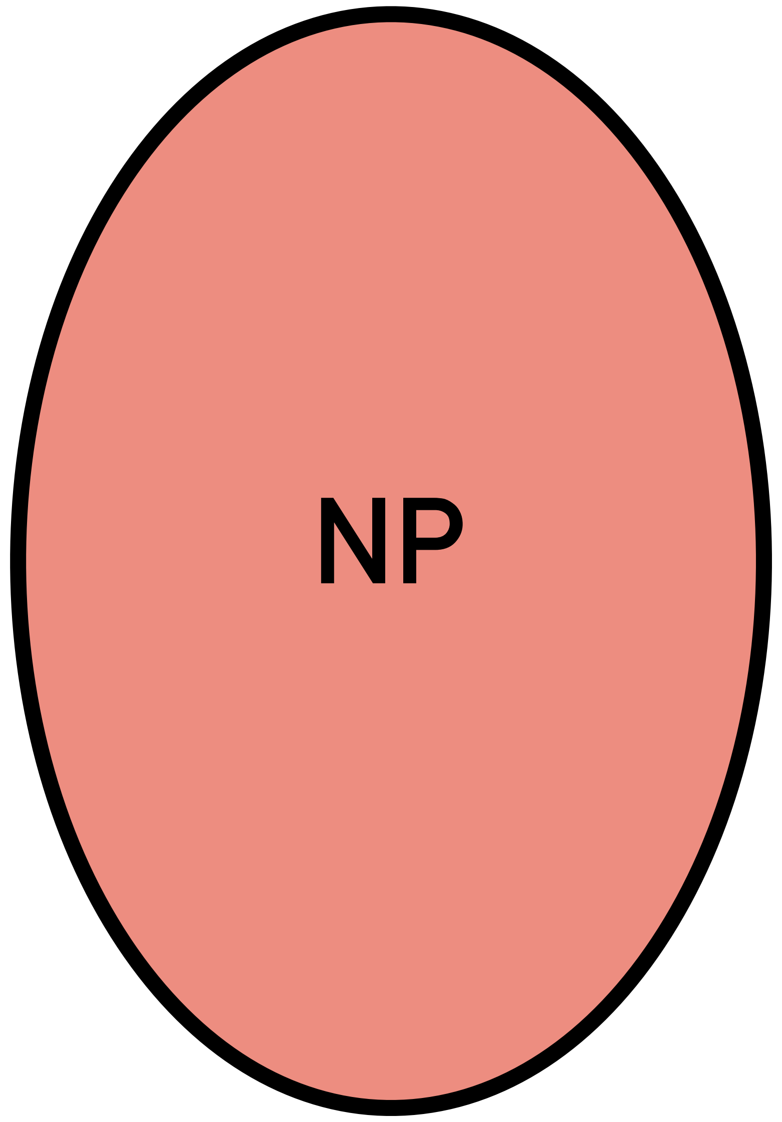

$\leq_p \; L \; \leq_p \; L'$

NP-Completeness and NP-Hardness

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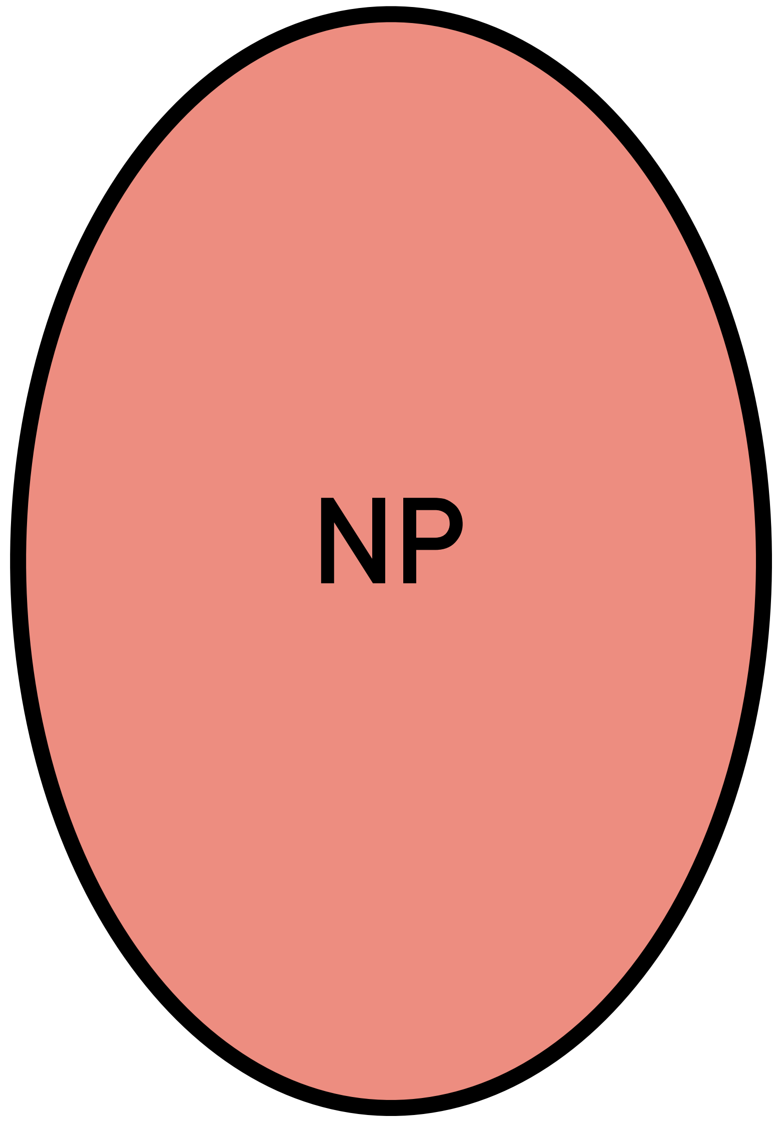

$\leq_p \quad L \quad \leq_p \quad L'$

Observation: If L is **NP-hard** and $L \leq_p L'$,

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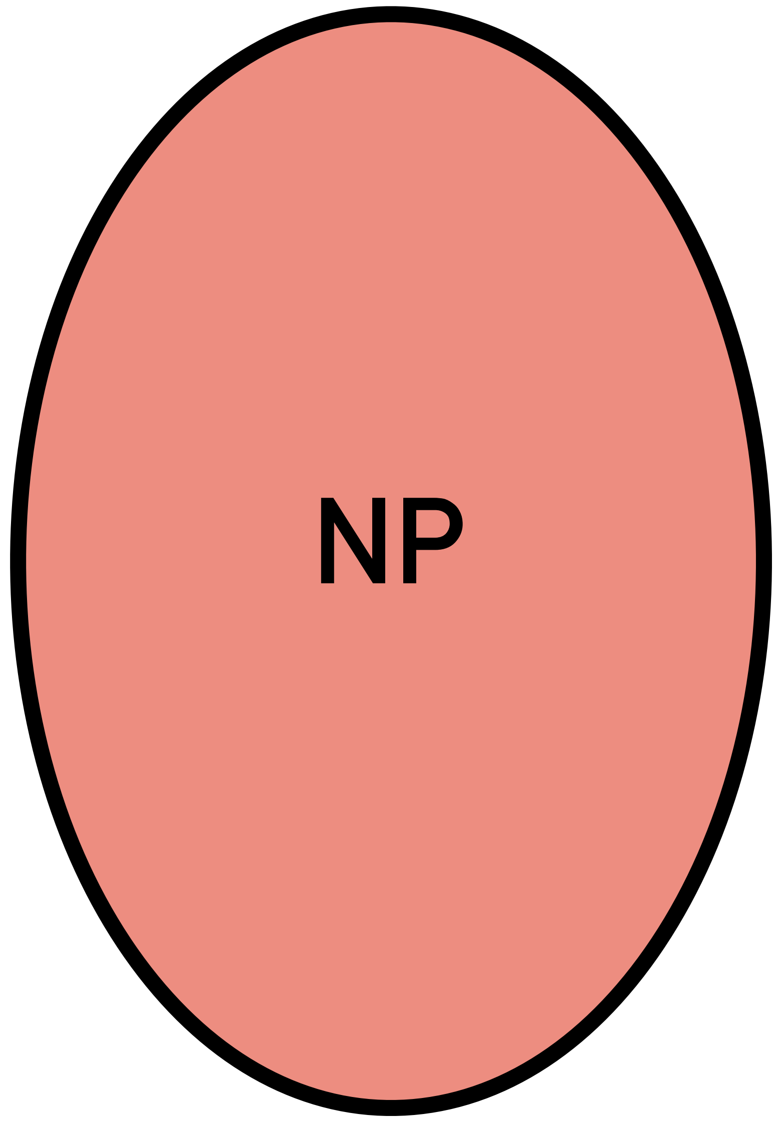


$\leq_p \; L \; \leq_p \; L'$

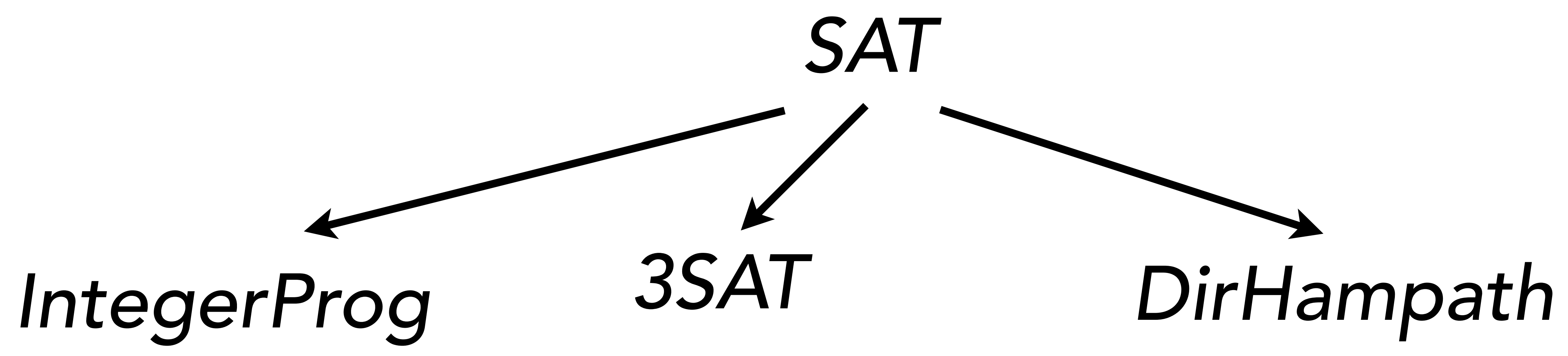
Observation: If L is **NP-hard** and $L \leq_p L'$, then L' is also **NP-hard**.

NP-complete **Network**

NP-complete **Network**

SAT

NP-complete **Network**



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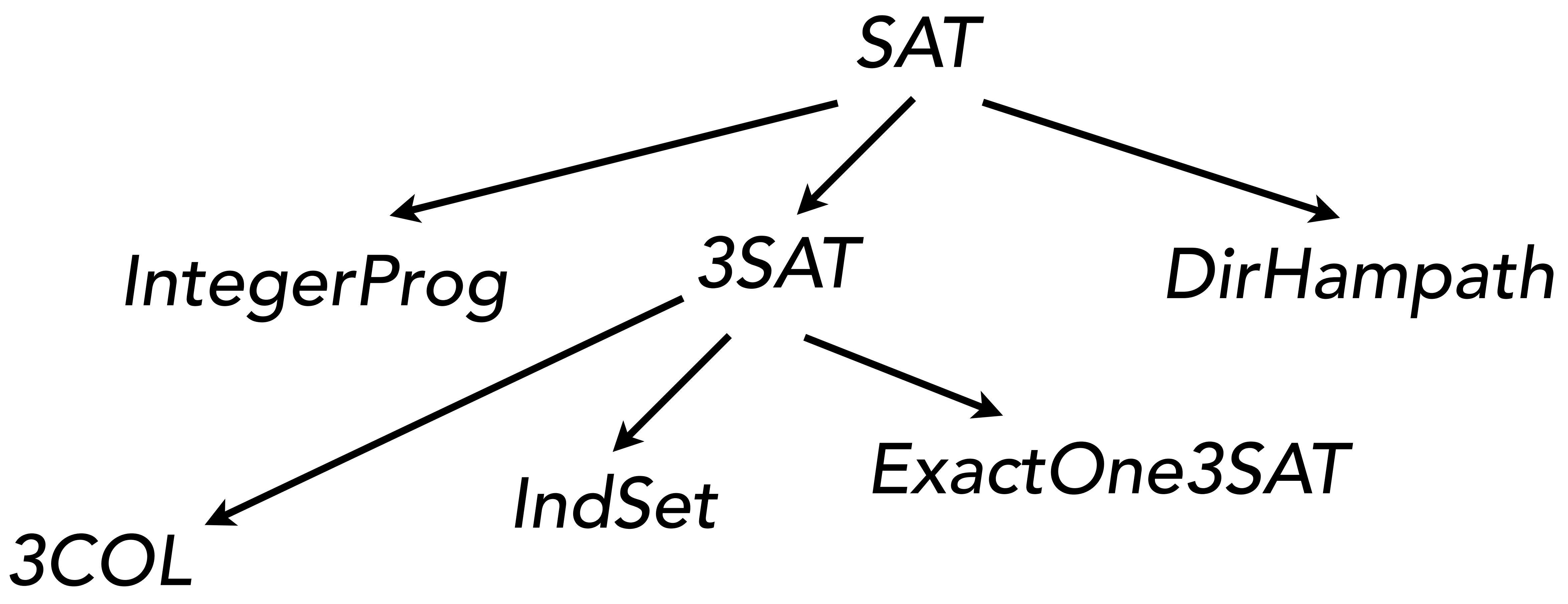

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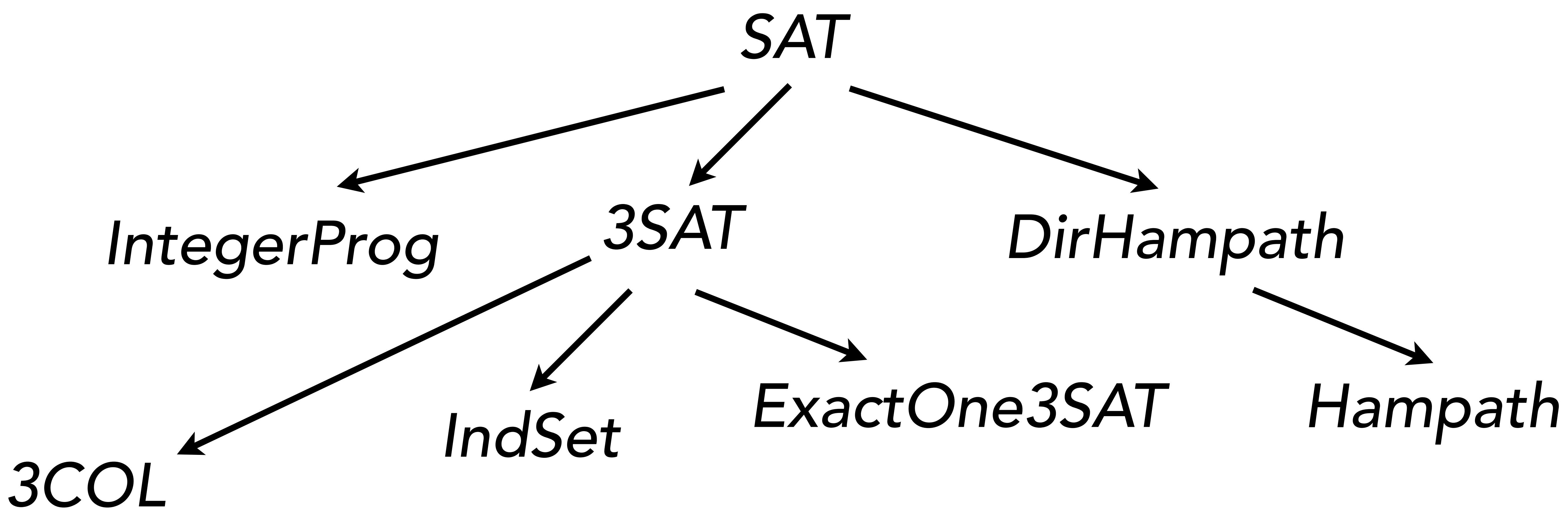

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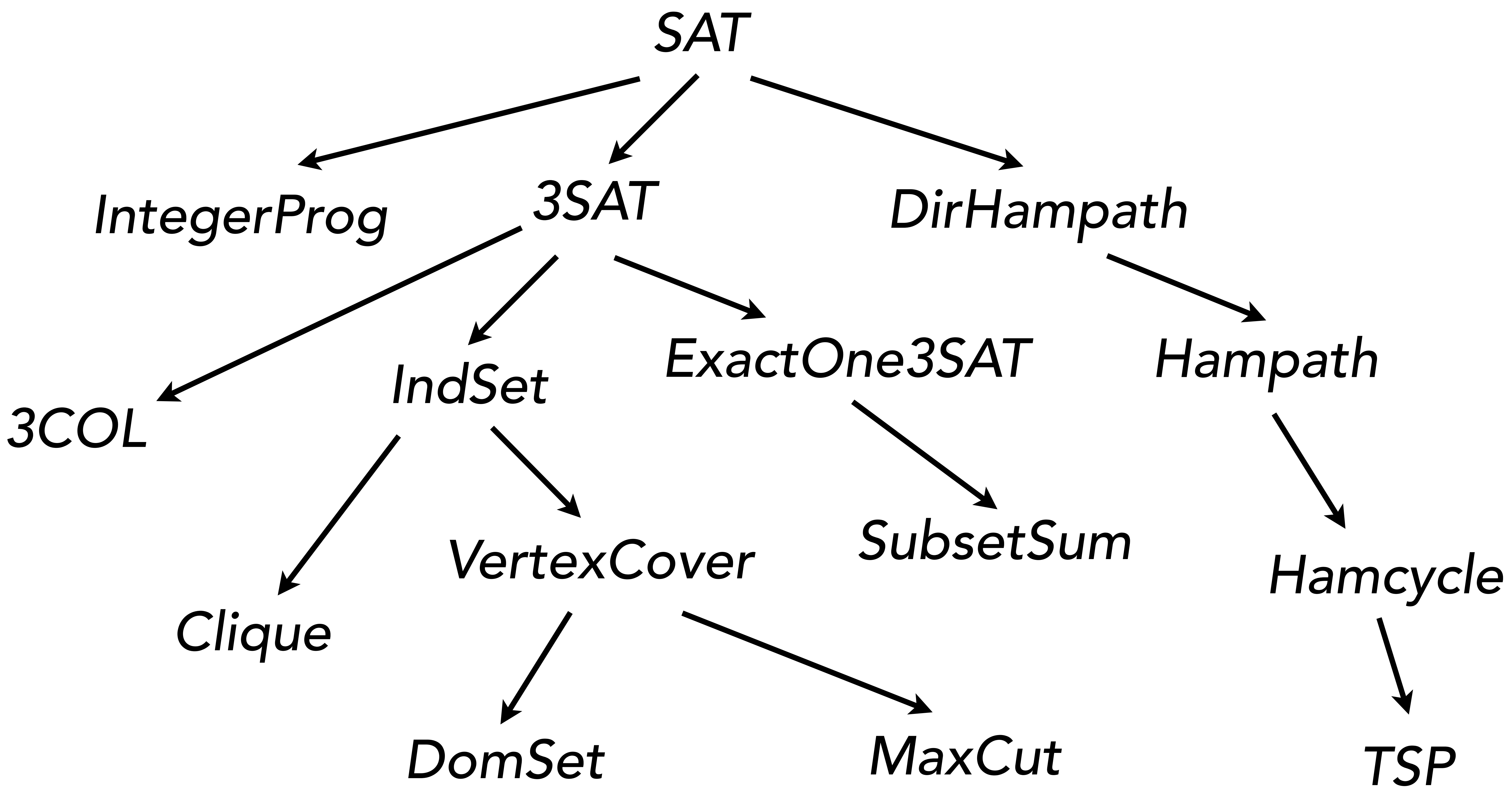

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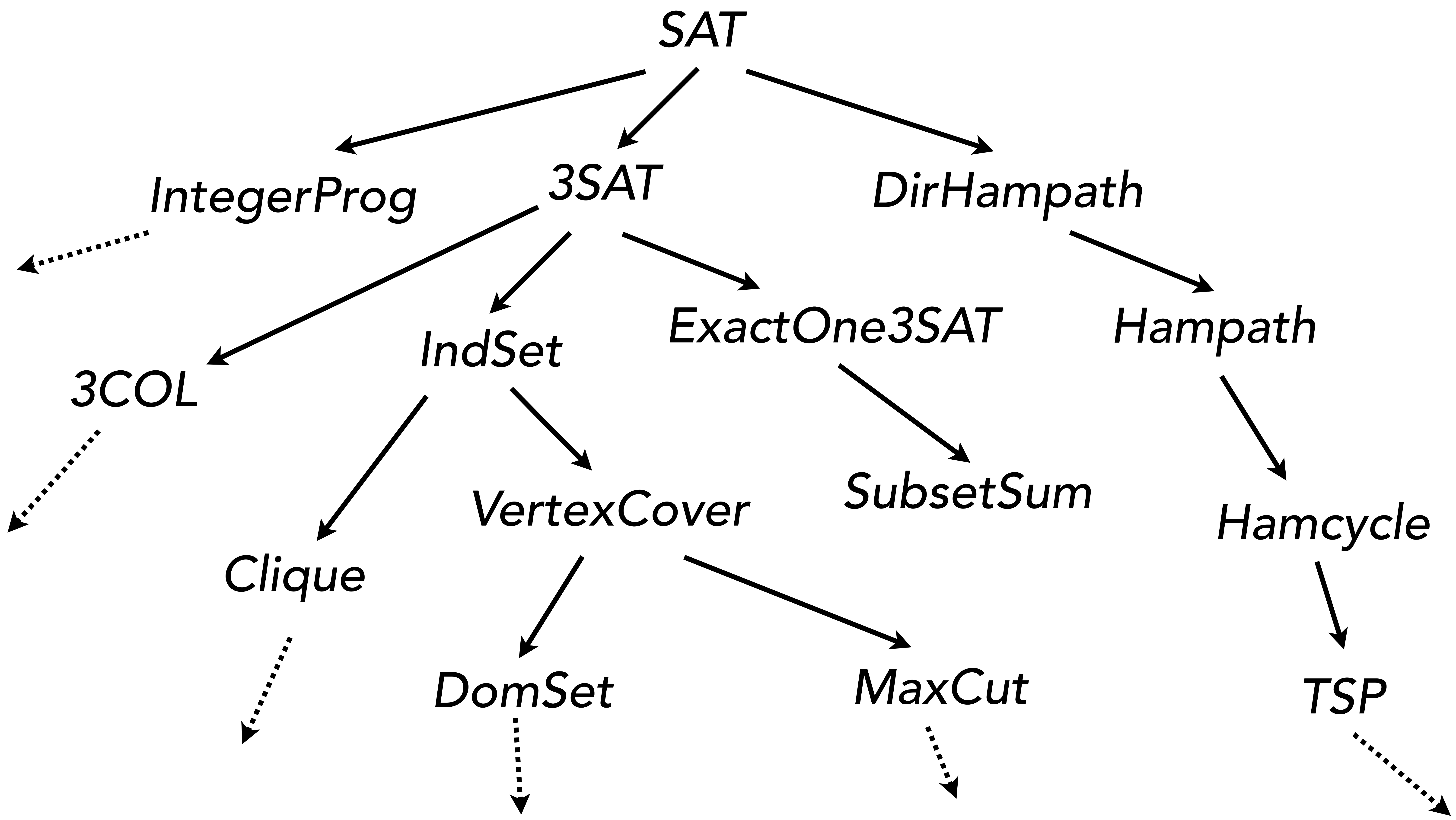

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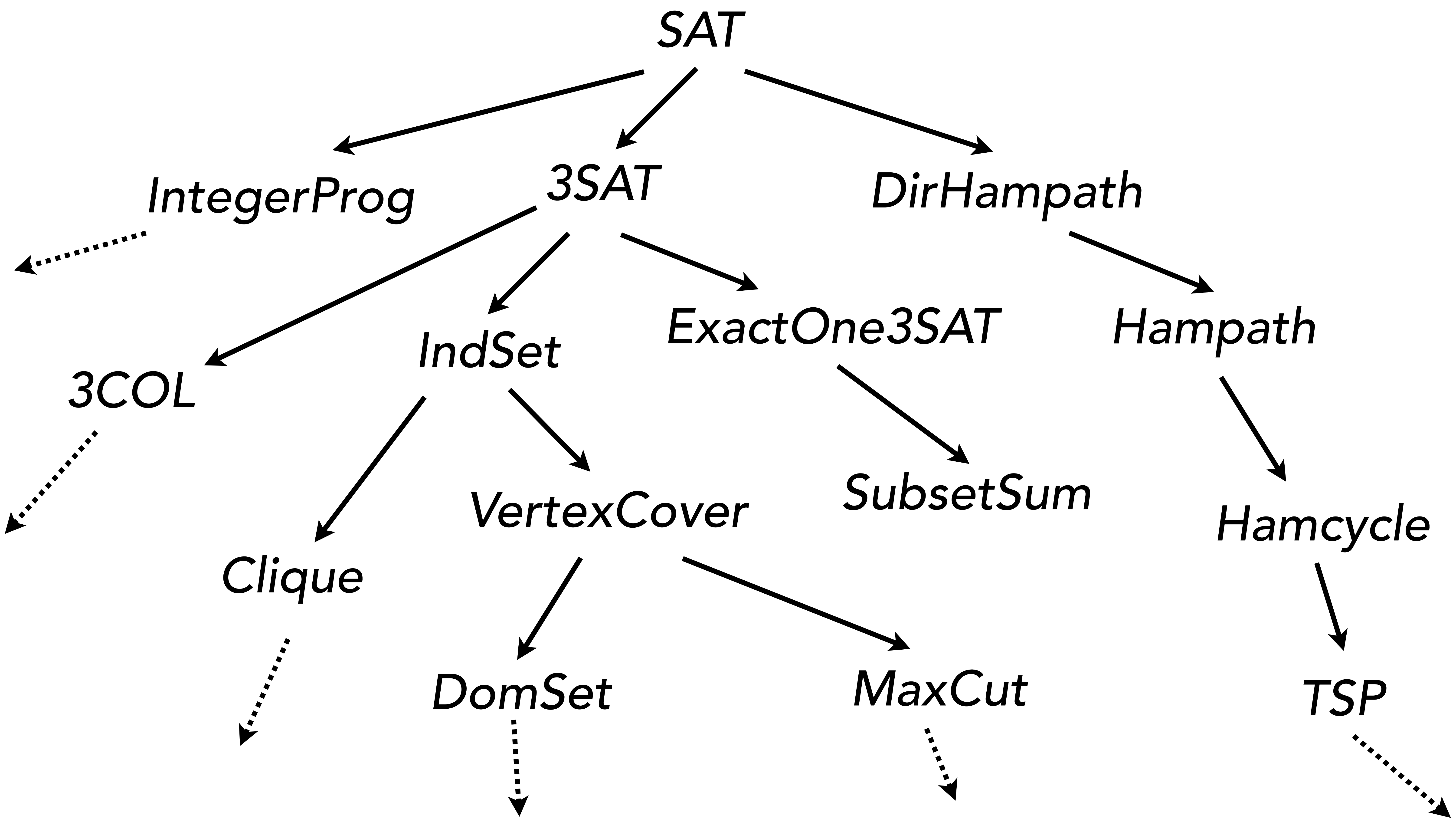

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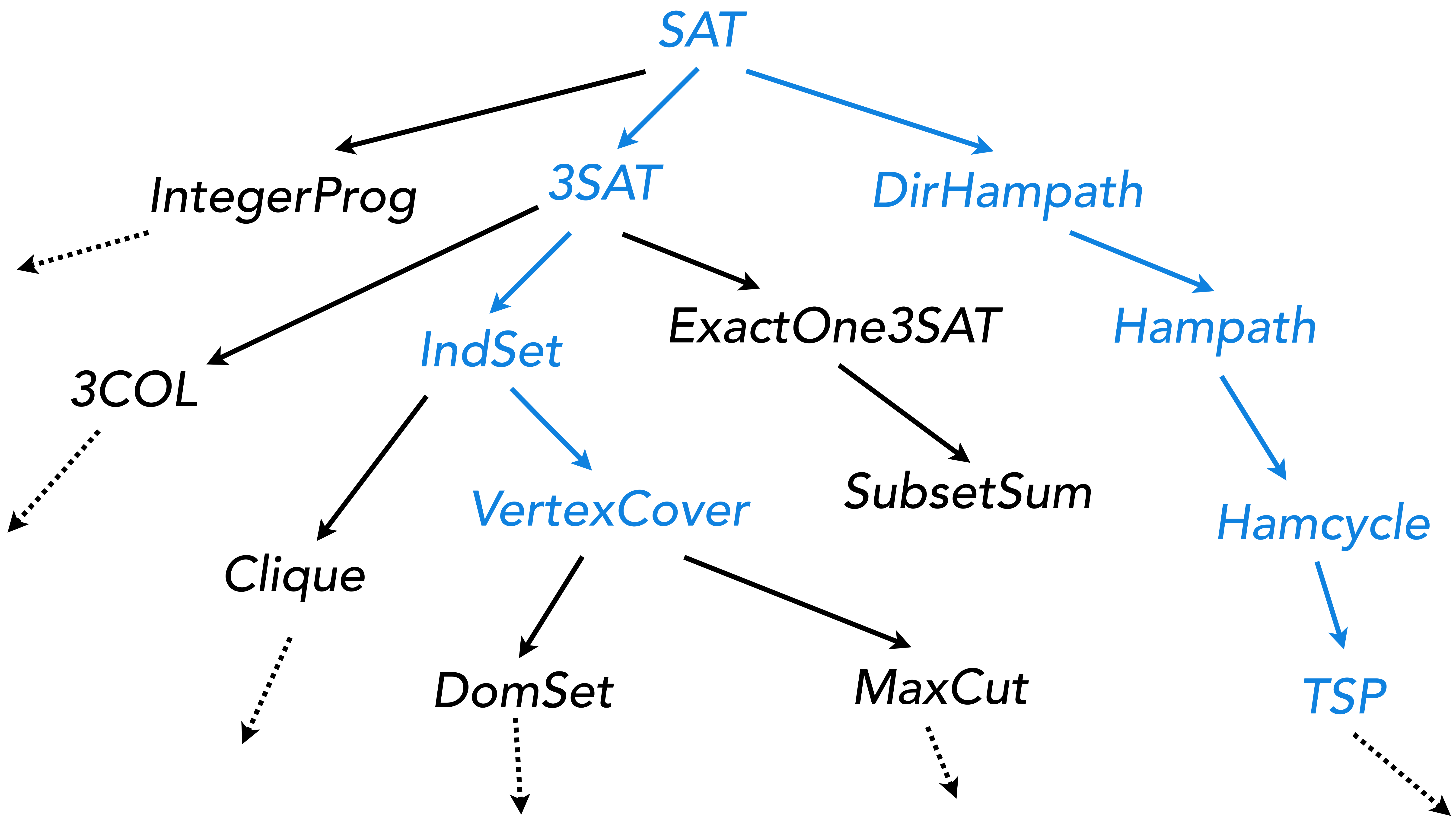

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Idea:

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$$\phi \;=$$

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Idea: Reduce *SAT* to *3SAT* by repeatedly breaking down clauses of $k > 3$ literals into two clauses of almost $k/2$ many literals such that the satisfiability is preserved.

$$\phi = (u_1 \vee u_2 \vee \ldots \vee u_k)$$

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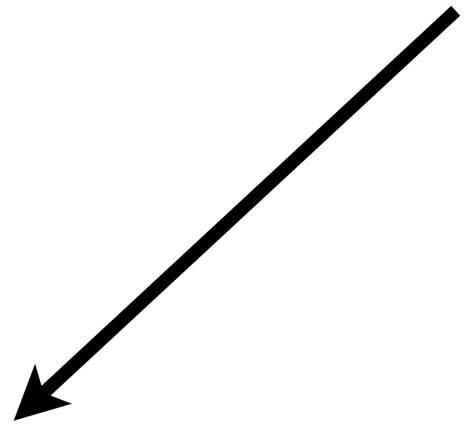

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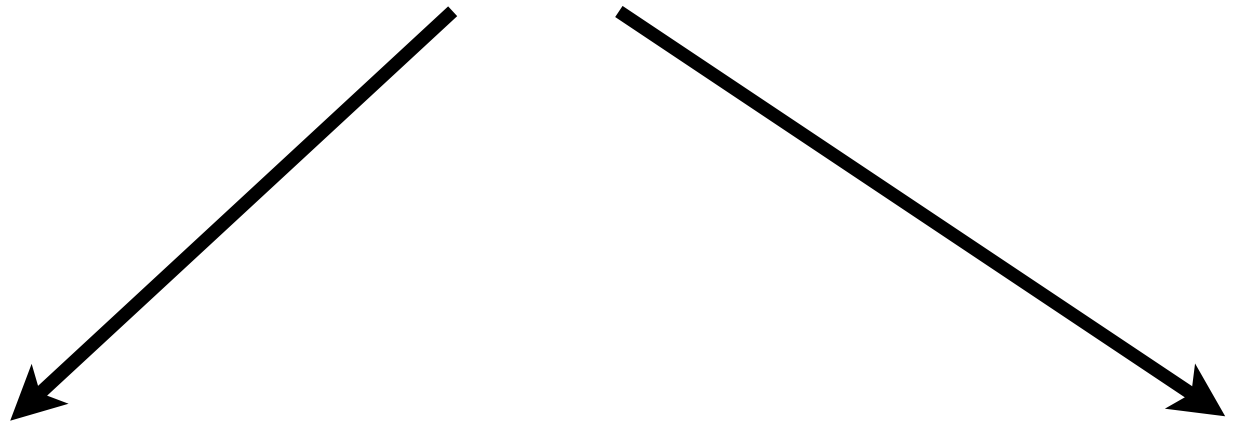

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Time to break a clause of k literals into a 3CNF formula:

- $T(k) = 2.T(k/2 + 1) + O(k)$
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- $T(k) = 2.T(k/2 + 1) + O(k)$
- $T(3) = c$

Prove that $T(k) = O(k^c)$ *and reduction is polytime.*

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Further breakdown isn't possible.

$3SAT \leq_p IndSet$

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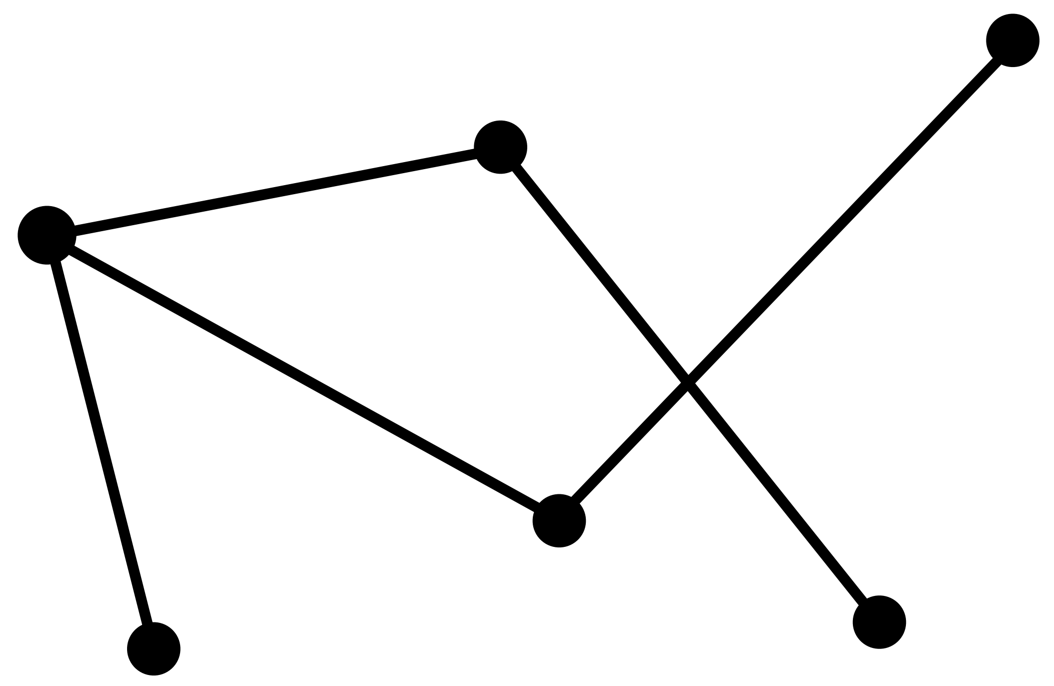

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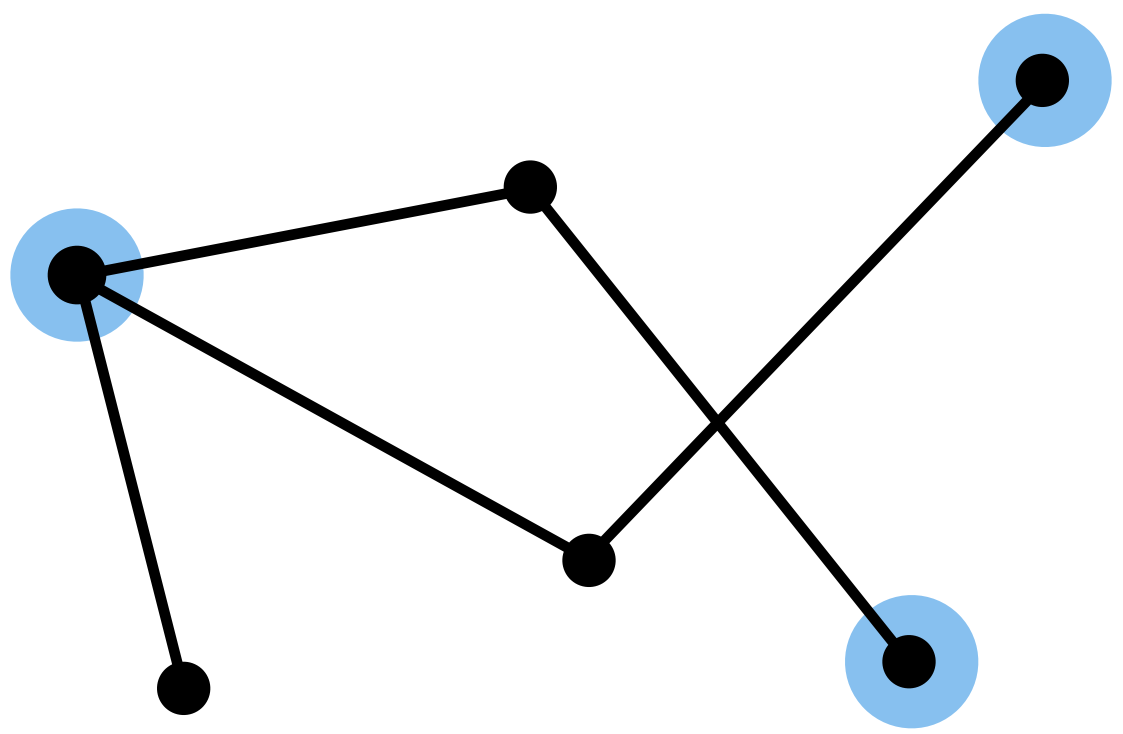

Has an independent set of size 3.

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v_1: 0 0 1

v_2: 0 1 0

v_3: 0 1 1

$(u_1 \vee u_2 \vee u_3) \implies$ v_4: 1 0 0

v_5: 1 0 1

v_6: 1 1 0

v_7: 1 1 1

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$(u_1 \vee u_2 \vee u_3) \implies$ v_4: 1 0 0 $(u_1 \vee \overline{u}_3 \vee u_4) \implies$

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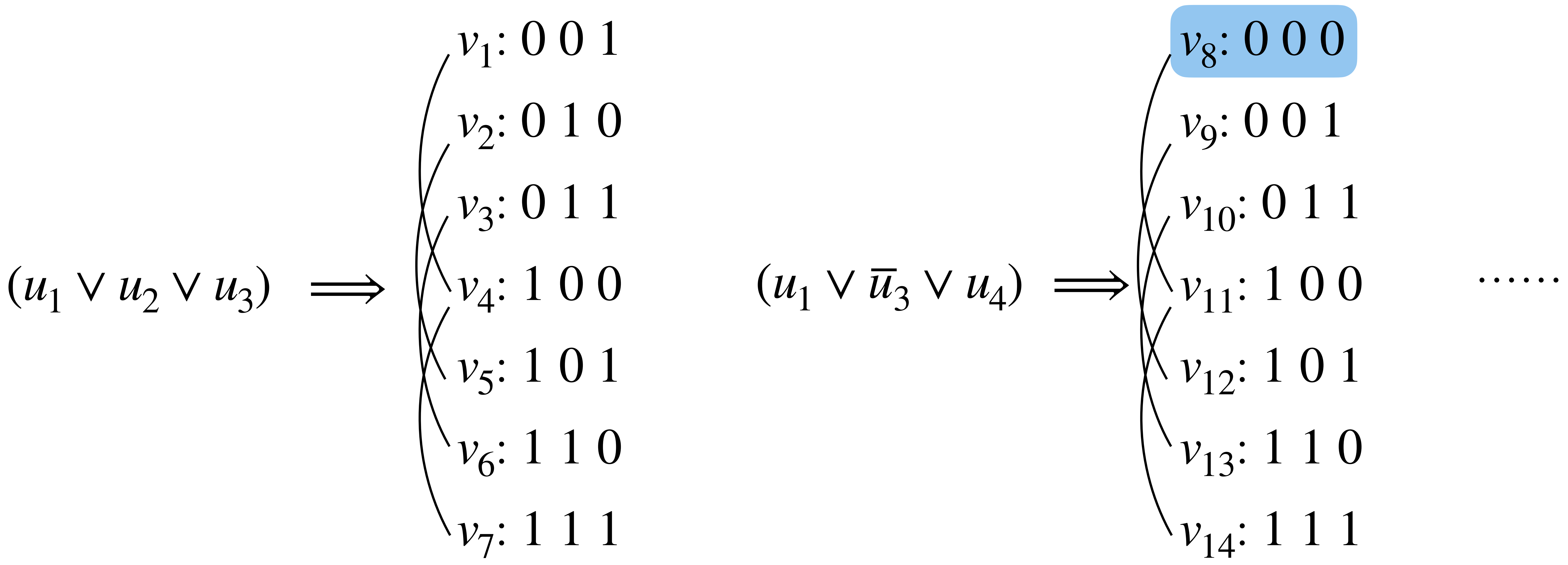

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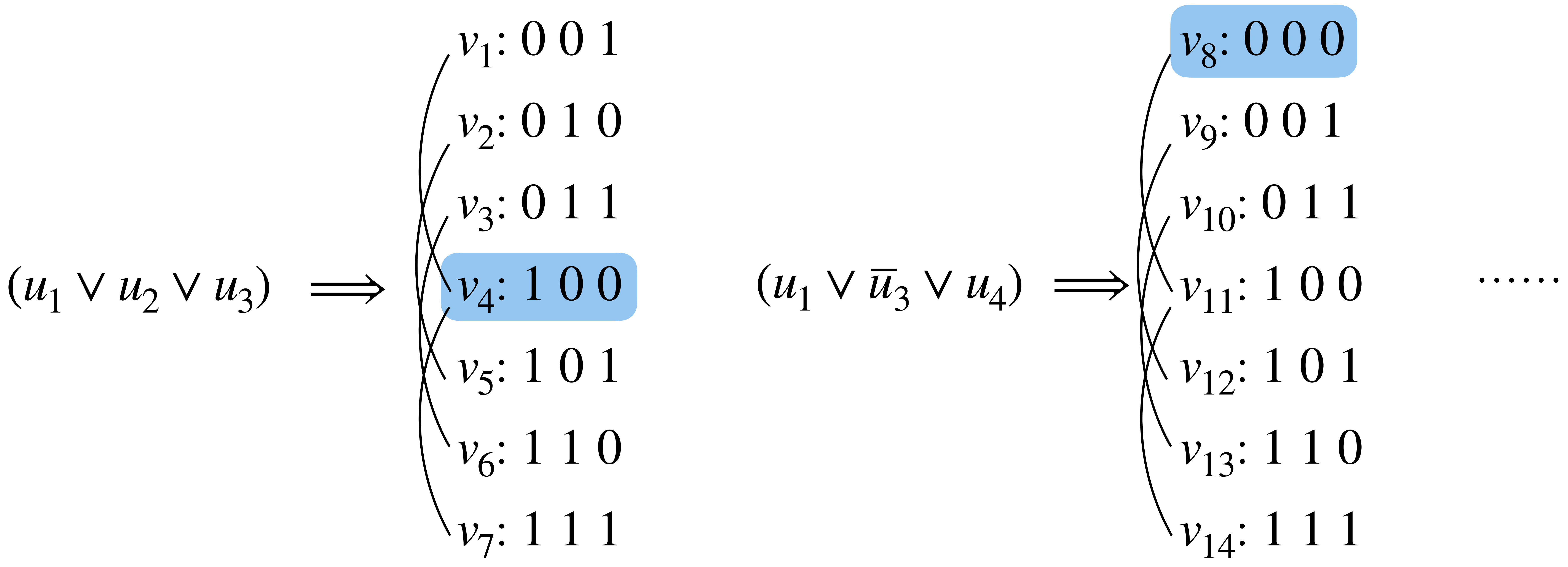

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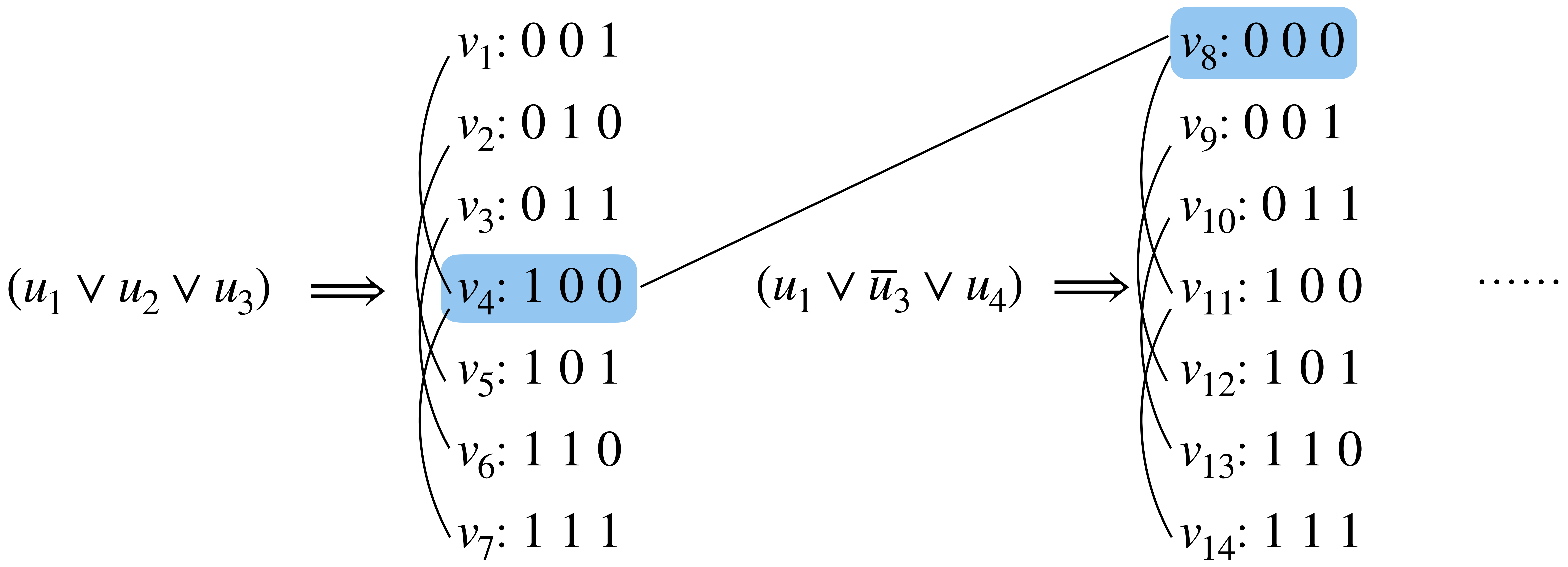

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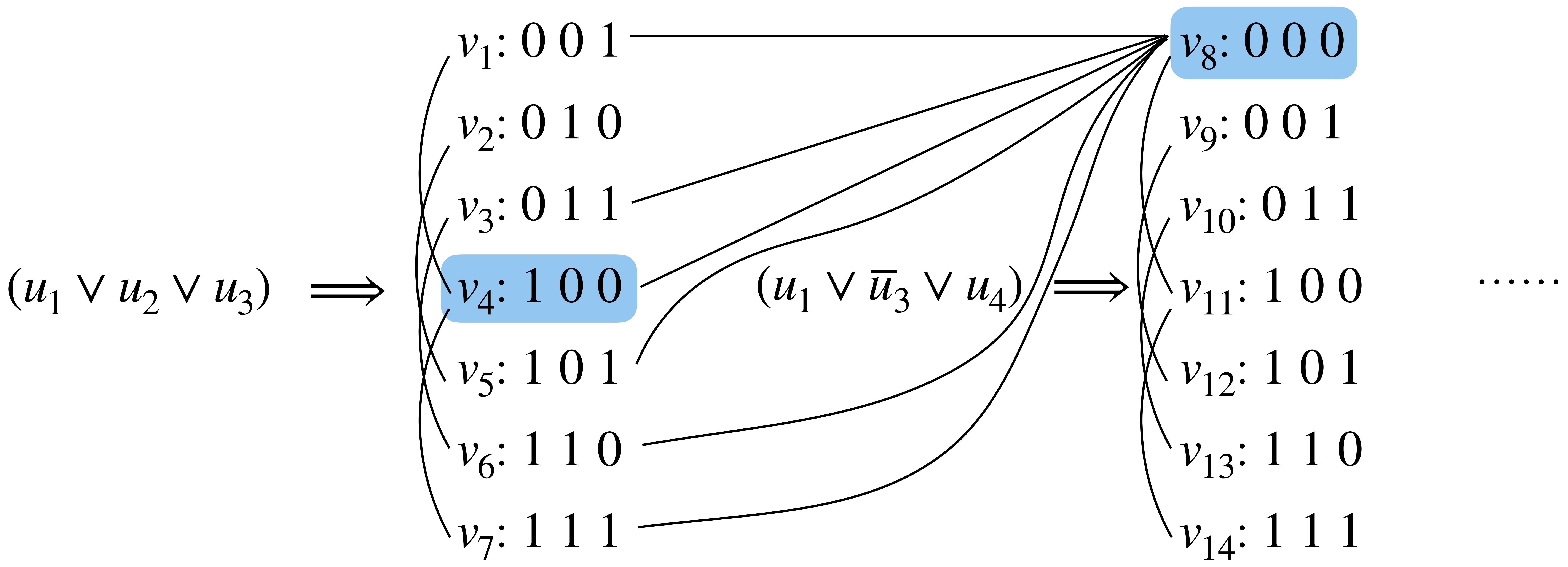

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$\phi \to \langle G, k \rangle$:

3SAT $\leq_p$ *IndSet*

$\phi \to \langle G, k \rangle$:

- A cluster of 7 vertices ∀ clause of ϕ corresponding to satisfying partial assignments.

$3SAT \leq_p IndSet$

$\phi \rightarrow \langle G, k \rangle$:

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$\phi \rightarrow \langle G, k \rangle$:

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- An edge between two vertices of different clusters, if they correspond to inconsistent partial assignments.

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Proof: ($\Longrightarrow$) Suppose ϕ has a satisfying assignment u.

$3SAT \leq_p IndSet$

$\phi \to \langle G, k \rangle$:

- A cluster of 7 vertices $\forall$ clause of ϕ corresponding to satisfying partial assignments.
- An edge between every pair of vertices in the same cluster.
- An edge between two vertices of different clusters, if they correspond to inconsistent partial assignments.
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By picking a vertex from every cluster whose values matches to that of u.